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Limit Lognormal Multifractal as an Exponential Functional

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The limiting distribution of the limit lognormal multifractal, first introduced by Mandelbrot (*Statistical Models and Turbulence*, M. Rosenblatt and C. Van Atta, eds., Lecture Notes in Physics **12**, Springer, New York, 1972, p. 333) and constructed explicitly by Bacry *et al.* (Phys. Rev. E **64**, 026103 (2001)), is investigated using its Laplace transform. A partial differential equation for the Laplace transform is derived and it is shown that multifractality alone does not determine the limiting distribution. The increments of the limit multifractal process are strongly stochastically dependent. The precise nature of this stochastic dependence structure of increments (SDSI) is the determining characteristic of the limiting distribution. The SDSI of the limit process is quantified by means of two integro-differential relations obtained by renormalization in the sense of Leipnik (J. Aust. Math. Soc. B **32**, 327–347 (1991)). One is interpreted as a counterpart of the star equation of Mandelbrot and the other is shown to be an analogue of the classical Girsanov theorem. In the weak intermittency limit an approximate single-variable equation for the Laplace transform is obtained and successfully tested numerically by simulation.

KEY WORDS: Limit lognormal; limit log-infinitely divisible; Mandelbrot star equation; Feynman-Kac formula; exponential functional; Brownian motion; Laplace transform.

1. INTRODUCTION

Following the seminal work of Mandelbrot[28] and Frisch and Parisi,[12] multifractals have been an integral part of various areas of physics such as turbulence,[11,34,41] stochastic growth,[15] percolation,[18] and geophysics[40] to name just a few. As Mandelbrot pointed out,[31] there are two

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distinct meanings of multifractality. The first meaning is that of interwoven fractal sets supporting different moments of a multifractal measure. This is the interpretation used in dynamical system theory(16,17) and, most recently, in the context of conformally invariant random paths or percolation.(8,9) The measures involved in such problems are typically nonrandom so that one can speak of fractal dimensions and a thermodynamic formalism(2,44). The second meaning of multifractality is related to the multiscaling of moments of random processes, such as the velocity field in fully developed turbulence(11,12,28) and asset prices in finance(37). The process is monofractal if its spectrum is linear as is the case of fractional brownian motion(33) and multifractal otherwise. Due to the stochastic nature of the problem, the thermodynamic formalism becomes much more intricate as the $f(\alpha)$ spectrum is dependent on a particular realization of the process and so is ill-defined in the usual sense. In the case of the canonical multifractals of Mandelbrot,(28) this difficulty was resolved by Molchan(35) in the physics and Arbeiter and Patzschke,(1) Barral(5) and others in the mathematics literature. We are primarily concerned with the multiscaling aspect of multifractals in this publication.

Multifractals are naturally classified as empirical and synthetic. Examples of empirical multifractals are numerous(18,26,34,37,43), whereas those of synthetic ones are very few. In fact, to the best of our knowledge, there are only four rigorous constructions. The first and most widely known class is that of the canonical multifractals of Mandelbrot.(28) This class includes nonrandom multifractal measures that occur in dynamical systems as a special case. Its most salient features are approximate multifractal scaling and discrete dilation invariance with a preferred scale that leads to nonstationary increments. These features are inextricably tied to the discrete cascade nature of the construction. The second and probably the oldest known construction is that of the limit lognormal multifractal, which was conceived by Mandelbrot(27,30) and formalized in a series of papers by Kahane(19–21). Its distinctive feature is that it is scale free, unlike the first class, and is defined as the zero scale limit over a set of dicrete scales. The first limit lognormal construction that did not involve discrete scales at all was given by Bacry *et al.*(3), who investigated many of the properties of the limit process without rigorously taking the zero scale limit.

While canonical and limit lognormal multifractals have been known for over twenty five years, it had remained a challenge to give a sound mathematical construction of scale free multifractals with continuous dilation invariance, stationary increments, and non-lognormal multipliers until Barral and Mandelbrot(6) made a breakthrough. Theirs is the third explicit construction. Their solution was to effect the idea of continuous multiplication by considering stochastic integrals over conical domains. A similar approach was taken by Schmitt and Marsan[42], who, however, did not take the zero-scale limit. The only limitations of the Barral and Mandelbrot solution were inexact scaling and log-compound-Poisson multipliers. The remaining problem was solved by Bacry and Muzy[36], who gave the fourth construction. Their approach, based on the previous work of Barral and Mandelbrot[6] and Schmitt and Marsan[42], was to change the shape of the conical domains of Barral and Mandelbrot. In addition, their construction allowed for arbitrary infinitely divisible multipliers and enabled accounting for a wide variety of possible spectra, including, in particular, the limit lognormal construction of Bacry *et al.*[3]. We will refer to these processes as the *limit log-infinitely divisible* multifractals.

The limit log-infinitely divisible process is defined as the zero scale limit of a series of finite scale processes. It is an open problem to mathematically describe the limiting distribution beyond its existence proven rigorously in the limit log-compound-Poisson case in ref. 6 and in general in ref. 4. Following the suggestion made in Muzy *et al.*[37] "to recast our approach within a field theoretical formulation involving some renormalization procedure", we will construct in this paper a framework for analyzing the finite-scale approximations. The framework provides a general approach to the study of stochastic processes that are built as exponential functionals, such as the limit log-infinitely divisible multifractals. First, the framework allows us to derive a Feynman-Kac formula for the exponential functional of a gaussian process, whether markovian or not. Specifically, it gives a partial differential equation for the Laplace transform of the limiting distribution of the limit lognormal multifractal process[27,3]. The equation shows that multifractality alone does not determine the distribution uniquely. Second, our framework imparts a precise meaning to the idea of Muzy *et al.*[37] of treating that distribution as a renormalized field theory. The concept of renormalization as formulated in this paper is an extension of the approach used by Leipnik[23] to investigate the lognormal distribution. Our extension provides a theoretical approach to quantifying the SDSI of the limit process, which is the main theme of this paper, and yields, in the lognormal case, a pair of exact integro-differential relations for the Laplace transform. These relations express the derivatives of the Laplace transform as appropriately rescaled Laplace transforms averaged over a family of limit lognormal processes derived from the limit lognormal multifractal. We propose to call such rescalings of the Laplace transform *renormalization in the sense of Leipnik*, who seems to have been the first to use this idea[23]. This is the main result of our paper.

The type of renormalization introduced is quite general as it applies to the exponential functional of general gaussian processes. In particular, the t relation yields a new proof of the classical Girsanov theorem for

Brownian motion and extends it to general gaussian processes. The z relation can be interpreted as a counterpart of the star equation of Mandelbrot[28,29] for the limit lognormal multifractal. It exhibits the fundamental characteristic of the star equation, namely, it is derived by taking the zero increment limit of a hierarchy of functional equations at finite increment sizes and thereby quantifies the SDSI of the limit process at a fixed time. The main difference between the two is that the star equation involves only one limit process, whereas our relation involves both the given limit lognormal process and a new family of such processes derived from it by a change of mean. This is an artifact of a more complicated SDSI involved in the scale invariant situation. The difference can be formally summarized by stating that our relation is not a single-variable equation. This is the main limitation of our work. In the weak intermittency limit, we give a single-variable equation for the Laplace transform. The equation is necessarily approximate yet it is in good agreement with results of numerical computation.

The plan of the paper is as follows. In Section 2 we give a review of the limit log-infinitely divisible processes. In Section 3 we describe the SDSI of the canonical multifractals of Mandelbrot in the context of the star equation and then explain how this SDSI is different from that of the limit log-infinitely divisible processes. In Section 4 an elementary derivation of our partial differential equation is presented based on multifractality alone. In Section 5 we introduce our framework and use it to rederive the equation and study its basic properties. Renormalization is defined in Section 6. In Section 7 we describe an explicit solution to the renormalization problem posed in Section 6. Section 8 gives an approximate treatment of the weak intermittency limit. Section 9 presents conclusions. In the Appendix we state and prove a generalized Girsanov theorem.

2. MULTIFRACTALS DEFINED

In this section we give a concise summary of the limit log-infinitely divisible processes following ref. 36. We start by describing their properties and then proceed to the constructive definition.

The limit log-infinitely divisible multifractal is a random process $X(t)$ defined as Brownian motion in multifractal time,

$$X(t) = B(M(t)), \tag{1}$$

an idea dating back to Mandelbrot and Taylor[32]. Multifractal time $M(t)$ is an increasing, positive random process independent of Brownian motion with the property that for any $0 < l < L$ and $0 < \gamma \leqslant 1$ its increments denoted by $\delta_l M(t) \equiv M(t+l) - M(t)$ obey the exact continuous cascade equation

$$\delta_{\gamma l} M(t) = W_\gamma \delta_l M(t) \tag{2}$$

and are stationary. W_γ is a positive stochastic multiplier independent of $\delta_l M(t)$ whose law must be log-infinitely divisible as was first pointed out by Novikov[38]. L is the fundamental or decorrelation scale of the process, which regulates its degree of long-range dependence. The multifractal spectrum ζ_q of $M(t)$ defined by the scaling of the qth absolute moment of the increments is set solely by the law of the multiplier W_γ. The cascade equation for $X(t)$ is

$$\delta_{\gamma l} X(t) = W_\gamma^{\frac{1}{2}} \delta_l X(t) \tag{3}$$

so that its spectrum is $\zeta_{q/2}$. A simple argument shows that at any given time t the distribution of $X(t)$ is related to that of $M(t)$ by

$$X(t) = \varepsilon M(t)^{\frac{1}{2}}, \tag{4}$$

where ε is a standardized normal random variable independent of $M(t)$. Moreover, Eq. (2) relates $M(t)$ to $M(L)$ for $t < L$, this relationship is known as Castaing's equation when expressed in terms of probability density functions[7]. Thus, to study $X(t)$ we may safely restrict ourselves to studying $M(L)$.

The formal mathematical construction of $M(t)$ starts with an infinitely divisible independently scattered random measure P on the time-scale plane (t, l), $l > 0$, distributed uniformly with respect to the measure μ

$$\mu(dt\,dl) = dt\,dl/l^2. \tag{5}$$

Infinite divisibility of P means that $P(A)$ is an infinitely divisible random variable for subsets A of the time-scale plane. Independent scatteredness means that $P(A)$ and $P(B)$ are independent if A and B do not intersect. Uniform distribution with respect to μ means that the characteristic function of $P(A)$ is given by

$$\mathbf{E}\left[e^{iqP(A)}\right] = e^{\phi(q)\mu(A)}, \tag{6}$$

where $\phi(q)$ is given by the Levy-Khinchine's formula[39]. We are primarily concerned with the case of a lognormal multiplier W_γ, in which case

$$\phi(q) = -i\, q\, \frac{\lambda^2}{2} - \frac{q^2 \lambda^2}{2}$$

for some $0 < \lambda < 1$, known as the intermittency parameter. The spectrum is always related to $\phi(q)$ via

$$\zeta(q) = q - \phi(-iq)$$

so that the lognormal spectrum is the familiar parabola[30]

$$\mathbf{E}\left[M(t)^q\right] = \left(\frac{t}{L}\right)^{\zeta_q} \mathbf{E}\left[M(L)^q\right], \quad t < L, \tag{7}$$

$$\zeta(q) = q\left(1 + \frac{\lambda^2}{2}\right) - \frac{\lambda^2 q^2}{2}. \tag{8}$$

The process $M(t)$ is defined as the zero scale limit $l \to 0$ of finite scale processes $M_l(t)$ that are themselves defined in terms of the measure P by

$$M_l(t) = \int_0^t e^{P(A_l(u))}\, du. \tag{9}$$

The sets $A_l(u)$ are sets in the time-scale plane whose precise definition given in ref. 36 does not concern us here. It is important to point out, however, that $A_l(u)$ and $A_l(v)$ intersect and so $P\,(A_l(u))$ and $P\,(A_l(v))$ are dependent whenever $|u - v| < L$. This explains the role of L as the measure of long-range dependence and accounts for most of the complexity of the construction. It is also important to point out that the distribution of $M_l(t)$, let alone

$$M(t) = \lim_{l \to 0} M_l(t), \tag{10}$$

is not known although the existence of the limit is established in ref. 4. The main tractable quantity is the characteristic function of the joint distribution of $P(A_l(t_1)), \ldots, P(A_l(t_N))$, $t_1 < \ldots < t_N$, denoted by $Q(\vec{t}, \vec{w})$

$$Q(\vec{t}, \vec{w}) \equiv \mathbf{E}\left[e^{i \sum_{j=1}^N w_j\, P(A_l(t_j))}\right]. \tag{11}$$

In fact, it is shown in ref. 36 that the scaling properties of $Q(\vec{t}, \vec{w})$ imply the existence of a random variable Ω_γ independent of $P(A_l(t))$ for all $0 \leqslant t \leqslant L$ such that

$$\left\{P(A_{\gamma l}(\gamma t))\right\} = \left\{\Omega_\gamma + P(A_l(t))\right\}, \ 0 \leqslant t \leqslant L, \tag{12}$$

understood as the equality in distribution of the two stochastic processes for fixed $0 < \gamma < 1$ and $0 < l < L$. It must be emphasized that Ω_γ is a true invariant as it is the same for all l and t. An elementary change of variables argument given in ref. 36 shows that Eq. (2) is a direct corollary of Eqs. (9) and (12) and

$$W_\gamma = \gamma e^{\Omega_\gamma}.$$

An explicit, while complicated, formula for $Q(\vec{t}, \vec{w})$ is derived in ref. 4. The formula simplifies significantly in the lognormal case because the joint distribution is normal and so can be described by its mean and covariance matrix.[2] These are

$$\mathbf{E}[P(A_l(t_i))] = -\frac{\lambda^2}{2}\left(1 + \log\frac{L}{l}\right), \tag{13}$$

$$\mathbf{Cov}\left[P(A_l(t_i)),\ P\left(A_l(t_j)\right)\right] = \lambda^2 \log\frac{L}{|t_i - t_j|},\ l \leqslant |t_i - t_j| \leqslant L, \tag{14}$$

$$\mathbf{Cov}\left[P(A_l(t_i)),\ P\left(A_l(t_j)\right)\right] = \lambda^2\left(1 + \log\frac{L}{l} - \frac{|t_i - t_j|}{l}\right), \tag{15}$$

if $|t_i - t_j| < l$, and covariance is zero in the remaining case of $|t_i - t_j| \geqslant L$. Thus, denoting the covariance matrix by C,

$$Q(\vec{t}, \vec{w}) = e^{-i\frac{\lambda^2}{2}(1+\log\frac{L}{l})\sum_{j=1}^N w_j}\, e^{-\frac{1}{2}\vec{w}^T C\, \vec{w}}. \tag{16}$$

This completes our overview of the limit log-infinitely divisible construction. The purpose of this paper is to describe the limiting distribution at time $t \leqslant L$ defined by Eq. (10) in a mathematical way. As it is shown in the two following sections, the fundamental structure of interest is the SDSI of the limit process.

[2] Note that the mean is half the negative variance. This is a necessary condition for the existence of the limit $l \to 0$ as first pointed out by Mandelbrot[(27)].

3. STOCHASTIC DEPENDENCE STRUCTURE OF INCREMENTS (SDSI)

In this section we compare and contrast the SDSI of the canonical multifractals of Mandelbrot(28,29) with that of the limit log-infinitely divisible processes(36). We also show that the star equation is a precise mathematical means of quantifying the SDSI in the canonical case.

The canonical construction of random multinomial measures begins with a fixed positive integer b and a positive random weight W such that $\mathbf{E}W = 1/b$. It consists of dividing the unit interval into b equal length subintervals, assigning them the random but stochastically independent measures distributed like W, then dividing each of the b subintervals into b pieces and assigning them measures as products of two independent weights distributed like W, and so on. We prefer to view the random measure constructed in this way as a stochastic process on the unit interval and refer to the cumulative weight of the subintervals as the increments of the process. Then, at the ith step of iteration the minimum size of $b-$adic[3] increments reached is $1/b^i$ and all the weights assigned at the ith step are independent, moreover, further subdivisions of these increments do not affect their dependence structure. Therefore, the $b-$adic increments of size $1/b^i$ share at most $i-1$ common random factors. In particular, they are independent when $i=1$. It then follows by self-similarity that the distribution of the limit process at time one, i.e., the limit measure of the unit interval, obeys the equation

$$Z = \sum_{i=0}^{b-1} W_j Z_j, \tag{17}$$

where W_j and Z_j are stochastically independent copies of W and Z, respectively. Equation (17) is known as the star equation of Mandelbrot. The formal proof of Eq. (17) is obtained by considering the $i \to \infty$ limit of a hierarchy of functional equations that express the measure of the unit interval at the ith step of iteration as a sum of the measures of all the $b-$adic subintervals of length b^{-i}. Therefore, it becomes transparent that Eq. (17) is a precise way of quantifying the SDSI of the limit process. This structure can be summarized in two statements: first, all increments that belong to the separate $b-$adic subdivisions of size $1/b$ are stochastically independent and, second, the dependence of all nonoverlapping $b-$adic size increments is through a *finite* product of independent positive weights.

[3] $b-$adic refers to the intervals of form $[kb^{-i}, (k+1)b^{-i}]$, where $i = 1, 2, 3 \ldots$ and $k = 0 \ldots b^i - 1$.

Naively, we can try to describe the SDSI of $M(t)$ in a similar way. By stationarity we can write an infinite sequence of functional equations, one for each positive integer b. Let $l_b = L/b$ and consider the increments $\delta_b(i) \triangleq M((i+1)l_b) - M(il_b)$, $i = 0 \cdots b-1$. All the $\delta_b(i)$ have the same distribution as $W_{1/b}M(L)$, i.e., are identically distributed. Obviously,

$$M(L) = \sum_{i=0}^{b-1} \delta_b(i).$$

At least formally, this system of equations, one for each b, resembles closely Eq. (17) in that both involve sums of length b such that the summands have the same distribution obtained by multiplying the unknown limiting distribution by a known positive random weight. However, the crucial difference between this system and Eq. (17) is that the summands in Eq. (17) are stochastically independent and the $\delta_b(i)$ are not. Moreover, *a priori*, we do not know how the $\delta_b(i)$ are dependent except by pointing at the definition, that is Eqs. (9) and (10). Comparing with the two properties of the canonical SDSI stated above, we have: first, *all* nonoverlapping increments that are within the distance L apart are dependent because $P(A_l(u))$ and $P(A_l(v))$ are whenever $|u-v| < L$ and, second, their dependence requires *infinitely* many independent factors because of an integral in Eq. (9). It is precisely these differences between the dependence structures that render scale invariant multifractality so complex. The precise goal of this paper is to quantify the SDSI of the limit lognormal process.

In the next section we show mathematically that multifractality alone does not capture the limiting distribution.

4. ELEMENTARY APPROACHES

In this section we give an elementary derivation of a partial differential equation for the Laplace transform of the limiting distribution of the limit lognormal multifractal[27,3]. It is based on Eq. (2), i.e., the continuous cascade equation.

We denote the Laplace transform by $v(t, z)$ so that

$$v(t, z) \equiv \mathbf{E}\left[e^{-zM(t)}\right], \; z \geqslant 0.$$

Then, the equation, which is the Feynman-Kac formula for the underlying gaussian process, is

$$z\frac{\partial v}{\partial z}=t\frac{\partial v}{\partial t}+\frac{\lambda^2}{2}z^2\frac{\partial^2 v}{\partial z^2},\ t\leqslant L,\ z\geqslant 0. \tag{18}$$

The boundary conditions for Eq. (18) are

$$v(t,\ z=0)=1 \text{ and } \lim_{z\to\infty}v(t,\ z)=0 \text{ for } t>0, \tag{19}$$

$$v(t=0,\ z)=1 \tag{20}$$

so we are dealing with a mixed initial/boundary value problem of parabolic type (to be solved backwards in time).

The proof of Eq. (18) is straightforward. By letting $\gamma\equiv 1-h/t$ in Eq. (2), we have

$$\frac{\partial v}{\partial t}(t,\ z)=-\lim_{h\to 0^+}\frac{\mathbf{E}\left[e^{-zW_\gamma M_t}\right]-v(t,\ z)}{h}.$$

It is easy to see based on results of[30,36] that $\log W_\gamma$ is gaussian with the mean $(1+\frac{\lambda^2}{2})\log\gamma$ and variance $-\lambda^2\log\gamma$. Since W_γ is independent of $M(t)$ in Eq. (2), we can reduce the expectation involved in two different ways resulting in two ways to complete the proof. Specifically,

$$\begin{aligned}\mathbf{E}\left[e^{-zW_\gamma M_t}\right] &= \mathbf{E}\left[v(t,\ zW_\gamma)\right],\\ &= \mathbf{E}\left[\psi(zM(t))\right],\end{aligned} \tag{21}$$

where $\psi(z)$ stands for the Laplace transform of W_γ evaluated at $zM(t)$. We choose the second equality as it naturally leads to the so-called Leipnik's equation that plays a major role in the subsequent analysis.

Let $\psi(z)$ stand for the Laplace transform of a lognormal random variable $Y\triangleq\exp(X)$ and let X have mean m and variance σ^2. Then, Leipnik[23] derived the following equation for $\psi(z)$

$$\frac{d}{dz}\psi(z)=-e^{m+\frac{\sigma^2}{2}}\psi(z\,e^{\sigma^2}),\ z\geqslant 0. \tag{22}$$

We will give another derivation in Section 6. Applying Leipnik's equation to W_γ, we get

$$\frac{d}{dz}\psi(z)=-\gamma\,\psi(z\,\gamma^{-\lambda^2}).$$

We are interested in the limit $\gamma \equiv 1 - h/t \to 1$. It is not hard to show that in this limit

$$\psi(z) = e^{-z}\left[1 + \frac{h}{t}\left(z + \frac{\lambda^2}{2}z^2\right)\right] + o(h).$$

Finally, substituting this expansion evaluated at $zM(t)$ into Eq. (21), we arrive at Eq. (18).

We end this section by discussing the informational content of Eq. (18). It is straightforward to check that $v(t, z)$ defined by

$$v(t, z) = \int_{\Re} \frac{F(ze^x)}{\sqrt{2\pi\lambda^2 \log\frac{T}{t}}} e^{-\frac{\left(x + (1+\frac{\lambda^2}{2})\log\frac{T}{t}\right)^2}{2\lambda^2 \log\frac{T}{t}}} dx \tag{23}$$

solves Eq. (18) for $t < T \leqslant L$ and satisfies the boundary conditions provided $F(z)$ is sufficiently smooth, $F(0) = 1$, and $F(z) \to 0$ as $z \to \infty$ fast enough. In particular, Eq. (18) alone does not determine $v(L, z)$.

The construction presented in this section shows that Eq. (18) is a necessary condition but is not sufficient to determine the limiting distribution of the limit lognormal multifractal. Thus, multifractality alone as understood in the sense of Eq. (2) does not determine the limiting distribution uniquely. Mathematically, this is atypical because, for example, the classical Feynman-Kac formula does determine the Laplace transform of the exponential functional of Brownian motion uniquely(10). On the other hand, the origin of non-uniqueness in Eq. (18) can be easily explained by noticing that Eq. (2) does not say anything about the SDSI of the limit process, which is the basic reason why Eq. (18) does not capture the limiting distribution.

In the remainder of this paper, we develop a framework to quantify the SDSI of the limit process culminating in a pair of renormalization relations in Section 7.

5. FINITE SCALE ANALYSIS

As we explained in Section 4 one needs to examine fine properties of $M(t)$ that go beyond its multifractality in the sense of Eq. (2). This section provides some first steps in this direction by introducing a novel discretization technique. The essence of our technique is to discretize the finite scale approximations in time and, thus, view them as finite sums of lognormal random variables. Such sum representations can then be naturally analyzed by means of the Laplace transform. In this section we describe the discretization procedure so as to set up the stage for renormalization analysis detailed in Sections 6 and 7. At the end of this section we make some remarks concerning Feynman-Kac formula for general gaussian processes.

We start with the definition of $M_l(t)$, i.e., Eq. (9). It says that $M_l(t)$ is the exponential functional of the underlying process $P(A_l(t))$. The theoretical foundation of our approach is the observation made in ref. 4 that one may discretize Eq. (9) as

$$M_{\Delta t}(t) \simeq \Delta t \sum_{j=1}^{N} e^{P(A_{\Delta t}(t_j))}, \quad \Delta t = \frac{t}{N}, \quad t_j = j\,\Delta t, \tag{24}$$

and the $N \to \infty$ limit coincides with $M(t)$. Denoting the Laplace transform of the discretized distribution by $v_{\Delta t}(t, z)$, one has by the definition

$$v_{\Delta t}(t, z) = \mathbf{E}\left[e^{-z\,M_{\Delta t}(t)}\right], \quad z \geqslant 0.$$

Then, $v_{\Delta t}(t, z)$ can be represented as a multiple countour integral

$$v_{\Delta t}(t, z) = \int_{\kappa - i\infty}^{\kappa + i\infty} L(\vec{t}, \vec{w})\,(z\,\Delta t)^{-\sum_{k=1}^{N} w_k} \prod_{j=1}^{N} \frac{\Gamma(w_j)}{2\pi i}\, d\vec{w}, \tag{25}$$

where κ is a large enough positive real number and $L(\vec{t}, \vec{w})$ is the Laplace transform of the joint distribution of $\{P(A_{\Delta t}(t_j))\}$. Equation (25) is the key to all subsequent developments. It must be emphasized that Eq. (25) is valid for all processes constructed as the exponential functional of an underlying process provided κ is positive and greater than the real part of all the singularities of $L(\vec{t}, \vec{w})$. If the underlying process is gaussian, then $L(\vec{t}, \vec{w})$ is related to the characteristic function $Q(\vec{t}, \vec{w})$ defined by Eq. (11) *via*

$$L(\vec{t}, \vec{w}) = Q(\vec{t}, i\vec{w}), \tag{26}$$

and κ can be any positive real number.

The proof of Eq. (25) is straightforward. The probability density function (pdf) of the joint distribution of $\{P(A_{\Delta t}(t_j))\}$ is given by the inverse Laplace transform

$$\frac{1}{(2\pi i)^N} \int_{\kappa - i\infty}^{\kappa + i\infty} L(\vec{t}, \vec{w}) e^{\vec{x}\cdot\vec{w}} d\vec{w} \tag{27}$$

given a large enough κ. It follows that the pdf of the joint distribution of $\{e^{P(A_{\Delta t}(t_j))}\}$ is given by

$$\frac{1}{(2\pi i)^N} \int_{\kappa - i\infty}^{\kappa + i\infty} L(\vec{t}, \vec{w}) x_1^{w_1 - 1} \cdots x_N^{w_N - 1} d\vec{w}. \tag{28}$$

The latter pdf enables us to write the Laplace transform of the sum that occurs in Eq. (24) as a multiple integral. Finally, we change the order of integration and use the gamma function identity

$$\int_0^\infty e^{-zx} x^{w-1} dx = z^{-w}\, \Gamma(w), \;\; z > 0, \;\; \Re w > 0.$$

In the lognormal case Eq. (25) can be reduced further so as to make explicit how $v_{\Delta t}(t, z)$ depends on z, t, and L. Since the finite scale l coincides with the time increment Δt, we can reduce the mean vector and covariance matrix of $P(A_l(t_j))$ given by Eqs. (13)–(15) to

$$\mathbf{E}[P\,(A_l(t_i))] = -\frac{\lambda^2}{2} \left(1 + \log N + \log \frac{L}{t}\right), \tag{29}$$

$$C_{ij} = \lambda^2 \left(\log \frac{N}{|i-j|} + \log \frac{L}{t}\right), \; i \neq j, \tag{30}$$

$$C_{ii} = \lambda^2 \left(1 + \log N + \log \frac{L}{t}\right). \tag{31}$$

Substituting these into Eq. (25), we find that the only dependence of the integrand of Eq. (25) on z, t, and L is in the multiplicative factor

$$\left(\frac{t}{L}\right)^{\zeta_{-\sum_{j=1}^N w_j}} (z\,L)^{-\sum_{k=1}^N w_k}, \tag{32}$$

where ζ_q is the multifractal spectrum as defined by Eq. (8).

An immediate conclusion to be drawn from Eq. (32) is that $v(t, z)$ is really a function of the variables t/L and zL. It is worth pointing out that Eq. (18) is also a direct corollary of Eq. (32). In fact, it is sufficient to show that Eq. (18) is satisfied by the function

$$(t, z, L) \mapsto \left(\frac{t}{L}\right)^{\zeta_q} (z\,L)^q$$

for each complex q, which is clear given the definition of ζ_q by Eq. (8).

We end this section with a comment regarding Feynman-Kac formula for general gaussian processes. We believe that the possibility to derive Eq. (18), which is nothing but a Feynman-Kac formula for the gaussian process involved, from Eq. (25) is by no means coincidental. Indeed, consider the limit

$$A \triangleq \lim_{\Delta t \to 0} \frac{\mathbf{E}\left[v_{\Delta t}\left(t,\, z e^{B(\Delta t)}\right)\right] - v_{\Delta t}(t,\, z)}{\Delta t}, \tag{33}$$

where $B(t)$ is standard Brownian motion. It is a classical result that

$$A = \frac{1}{2}\left[z\frac{\partial v}{\partial z} + z^2\frac{\partial^2 v}{\partial z^2}\right]. \tag{34}$$

On the other hand, the limit can be expressed as a combination of $v(t,\, z)$ and its derivatives in t and z using Eq. (25). That the t derivative must be involved is clear from the fact that $\mathbf{E}\left[v_{\Delta t}\left(t,\, z e^{B(\Delta t)}\right)\right]$ brings out the extra factor of $e^{\frac{\Delta t}{2}(\sum_{j=1}^{N} w_j)^2}$ in the integrand of Eq. (25). Since this factor is quadratic in the $w_j s$, and the only other term in the integrand quadratic in the $w_j s$ is $L(\vec{t}, \vec{w})$, this is equivalent to changing the covariance matrix of the process that depends on t. This is how one can derive Feynman-Kac formula for general gaussian processes, at least in principle. It is easy to see that the described recipe yields an elementary proof of the classical Feynman-Kac formula for Brownian motion.

In the next section we will extend the range of applications of our finite scale analysis beyond the Feynman-Kac formula.

6. RENORMALIZATION: DEFINITIONS

As we explained in Section 4, Eq. (2), i.e., the continuous cascade equation, does not determine the limiting distribution of the limit lognormal multifractal. The missing piece of information needed is the SDSI of the process. The goal of this and the following section is to quantify this structure by means of renormalization in the sense of Leipnik[23] defined in the Introduction to mean relations expressing $\partial v/\partial z$ and $\partial v/\partial t$ as the rescaled Laplace transform averaged over a family of limit lognormal multifractals.

Our approach is to quantify the structure of dependence by analyzing the Laplace transform of $M(t)$ and specifically Eq. (25). The main idea of the discretization technique introduced in the preceding section was to approximate $M_l(t)$ by a sum of lognormal random variables resulting in

Eq. (25). In this section we use that approximation to extend Leipnik's characterization of the lognormal distribution given by Eq. (22) to $M_l(t)$. The essence of Eq. (22) is that the derivative of the Laplace transform equals the product of a weight factor and the rescaled Laplace transform. We show below that one can extend this result to $M_l(t)$ and eventually to $M(t)$.

We begin by giving a derivation of Eq. (22). As in section 4 let $\psi(z)$ stand for the Laplace transform of a lognormal random variable $Y \triangleq \exp(X)$ and let X have mean m and variance σ^2. Leipnik's original proof used explicitly the special properties of lognormal density function. Here we give an alternative proof that can be automatically extended to $M_l(t)$ later. We start with Eq. (25) for $N=1$

$$\psi(z) = \int_{\kappa-i\infty}^{\kappa+i\infty} e^{\frac{1}{2}w^2\sigma^2 - mw}\, z^{-w} \frac{\Gamma(w)}{2\pi i}\, dw. \tag{35}$$

Differentiating with respect to z, making use of the functional equation of the gamma function

$$w\,\Gamma(w) = \Gamma(w+1),\ \ \Re(w) > 0,$$

and changing variables $w' = w+1$, we get

$$\frac{d}{dz}\psi(z) = -e^{m+\frac{\sigma^2}{2}} \int_{\kappa-i\infty}^{\kappa+i\infty} e^{\frac{1}{2}w'^2\sigma^2 - mw'}\, (ze^{\sigma^2})^{-w'} \frac{\Gamma(w')}{2\pi i}\, dw'.$$

Recalling Eq. (35), we arrive at Leipnik's equation. Effectively, we have renormalized Eq. (35) by expressing the derivative of the Laplace transform as an appropriately rescaled Laplace transform. The purpose of renormalization in general is to do the same with Eq. (25) in the limit $N \to \infty$ with respect to both z and t.

The multifractal analogue of Leipnik's equation can be obtained from Eq. (25) by following the same steps. From now on we drop the limits of integration for typographic simplicity. By default these limits are same as in Eq. (25).

$$\frac{\partial}{\partial z} v_{\Delta t}(t, z) = -\sum_{k=1}^{N} \Delta t \int L(\vec{t}, \vec{w} - \vec{k})\, (z\,\Delta t)^{-\sum_{m=1}^{N} w_m} \prod_{j=1}^{N} \frac{\Gamma(w_j)}{2\pi i}\, d\vec{w}. \tag{36}$$

Here $\vec{k}$ is the vector whose only nonzero component is the kth component, which equals 1. The validity of Eq. (36) is quite general as it applies to all limit log-infinitely divisible processes. In essence Eq. (36) reduces the complexity of the whole problem to the task of analyzing $L(\vec{t},\ \vec{w})$. As long as the latter can be computed as the analytic continuation of $Q(\vec{t}, i\vec{w})$ given by Eq. (26), we have effectively reduced the problem to analyzing the main tractable quantity available as pointed out in Section 2. In the lognormal case Eq. (26) holds true so that we have an explicit formula for $L(\vec{t},\ \vec{w})$ and can reduce Eq. (36) further. This is due to

$$L(\vec{t},\ \vec{w}-\vec{k}) = L(\vec{t},\ \vec{w})\, e^{-\vec{w}\cdot\vec{c}_k}, \tag{37}$$

where $\vec{c}_k$ is the kth column of the covariance matrix C stated in Eqs. (30) and (31). It must be pointed out that Eq. (37) holds only for the special choice of the mean as in Eq. (13). Otherwise, there would be an extra factor that blows up in the limit $N\to\infty$. The finite scale equation then becomes

$$\frac{\partial}{\partial z} v_{\Delta t}(t,\ z) = -\sum_{k=1}^{N} \Delta t \int L(\vec{t},\ \vec{w})\, e^{-\vec{w}\cdot\vec{c}_k}\, (z\,\Delta t)^{-\sum_{m=1}^{N} w_m} \prod_{j=1}^{N} \frac{\Gamma(w_j)}{2\pi i}\, d\vec{w}. \tag{38}$$

Thus, the goal of renormalization with respect to z in the lognormal case is to take the $N\to\infty$ limit of Eq. (38) so as to re-express the right-hand side as an average of rescaled Laplace transforms.

Renormalization with respect to t is defined in the same way. A straightforward computation yields

$$\frac{\partial v_{\Delta t}}{\partial t} = -z\,\mathbf{E}\left[e^{-zM_{\Delta t}(t)} e^{P(A_{\Delta t}(t_N))}\right], \tag{39}$$

$$= -z \int L(\vec{t},\ \vec{w})\, e^{-\vec{w}\cdot\vec{c}_N}\, (z\,\Delta t)^{-\sum_{m=1}^{N} w_m} \prod_{j=1}^{N} \frac{\Gamma(w_j)}{2\pi i}\, d\vec{w}. \tag{40}$$

The similarity of Eqs. (38) and (40) is clear. The source of difficulty in both is the same, namely, the occurrence of $e^{-\vec{w}\cdot\vec{c}_k}$ factors.

In the next section, we will evaluate and analyze the $\Delta t\to 0$ limit of Eqs. (38) and (40).

7. RENORMALIZATION: EXACT RESULTS

We defined the notion of renormalization in Section 6 as a means of taking the $\Delta t\to 0$ limit of Eqs. (38) and (40) so as to re-express the partials on the left-hand side as rescaled Laplace transforms. It turns out that the right-hand sides of Eqs. (38) and (40) can indeed be interpreted in this limit as averages of rescaled Laplace transforms, albeit of different but related processes. These new processes form a family of limit lognormal multifractals, however, they do not possess stationary increments. As we show below this type of renormalization is quite general as it applies to all gaussian processes, not just limit lognormals and in that sense is a true extension of the original Leipnik's treatment of lognormals[23].

We start by constructing the new family of limit lognormal multifractals first. It needs to be clarified that multifractality is understood here strictly in the sense of Equation (7), i.e., multiscaling. Let the lognormal stochastic measure P be the same as defined in Section 2 and for each $0 \leqslant u \leqslant 1$ define

$$M_l^u(t) = \int_0^t e^{P(A_l(s))} \left|\frac{s}{t} - u\right|^{-\lambda^2} ds, \tag{41}$$

$$M^u(t) = \lim_{l\to 0} M_l^u(t). \tag{42}$$

Since $0 < \lambda < 1$, the singularity is integrable and the limit exists. It is straightforward to show that $M^u(t)$ is multifractal for each $0 \leqslant u \leqslant 1$. Indeed, given $0 < \gamma < 1$ and $0 < l < L$, consider $M^u_{\gamma l}(\gamma t)$ and change variables $s = \gamma s'$ in Eq. (41)

$$\begin{aligned} M^u_{\gamma l}(\gamma t) &= \int_0^{\gamma t} e^{P(A_{\gamma l}(s))} \left|\frac{s}{\gamma t} - u\right|^{-\lambda^2} ds, \\ &= \gamma \int_0^t e^{P(A_{\gamma l}(\gamma s'))} \left|\frac{s'}{t} - u\right|^{-\lambda^2} ds'. \end{aligned}$$

Now, by Eqs. (12) and (42) we can write

$$\begin{aligned} M^u(\gamma t) &= \lim_{l\to 0} M^u_{\gamma l}(\gamma t), \\ &= \gamma e^{\Omega_\gamma} \lim_{l\to 0} \int_0^t e^{P(A_l(s'))} \left|\frac{s'}{t} - u\right|^{-\lambda^2} ds', \\ &\equiv W_\gamma \, M^u(t). \end{aligned}$$

This argument is patterned on the original proof of multifractality of $M(t)$ given in ref. 36.

We now proceed to take the $\Delta t \to 0$ limit of Eqs. (38) and (40). The key observation is that the $e^{-\vec{w}\cdot\vec{c}_k}$ factors can be interpreted as a change of mean. Specifically, the mean of the gaussian vector

$$P(A_l(t_1)), \ldots, P(A_l(t_N)), \; t_1 < \cdots < t_N,$$

is defined by Eq. (13). The Laplace transform of a general gaussian vector with covariance matrix C and mean vector $\vec{m}$ is seen from Eqs. (16) and (26) to be

$$L(\vec{w}) = e^{-\vec{m}\cdot\vec{w}}\, e^{\frac{1}{2}\vec{w}^T C\, \vec{w}}.$$

Thus, multiplying $L(\vec{t}, \vec{w})$ by $e^{-\vec{w}\cdot\vec{c}_k}$ is equivalent to adding $\vec{c}_k$ to the mean of $P(A_l(t_1)), \ldots, P(A_l(t_N))$. By following the steps that led to Eq. (25) backwards, we may thus re-express the summands that occur on the right-hand side of Eq. (38) as

$$\mathbf{E}\left[e^{-z\,\Delta t \sum_{j=1}^{N} e^{P(A_{\Delta t}(t_j))+C_{jk}}}\right].$$

Next, recalling that $t = N\,\Delta t$, converting all sums into integrals in the limit $\Delta t \to 0$, and making use of Eqs. (9) and (30), we reduce Eq. (38) to

$$\frac{\partial}{\partial z} v(t, z) = -t \lim_{\Delta t \to 0} \int_0^1 \mathbf{E}\left[e^{-z\left(\frac{t}{L}\right)^{-\lambda^2} \int_0^t \left|\frac{s}{t}-u\right|^{-\lambda^2} dM_{\Delta t}(s)}\right] du.$$

It remains to recall the definition of $M^u(t)$ to finally write

$$\frac{\partial}{\partial z} v(t, z) = -t \lim_{\Delta t \to 0} \int_0^1 \mathbf{E}\left[e^{-z\left(\frac{t}{L}\right)^{-\lambda^2} \int_0^t dM^u_{\Delta t}(s)}\right] du, \tag{43}$$

$$= -t \int_0^1 \mathbf{E}\left[e^{-z\left(\frac{t}{L}\right)^{-\lambda^2} M^u(t)}\right] du. \tag{44}$$

We have thus established a precise link between the limit lognormal process and those introduced above. As expected the z derivative of $v(t, z)$ is equal to an average of rescaled Laplace transforms, those of $M^u(t)$. Since $M^u(t)$ can be formally expressed in terms of $M(t)$ as

$$M^u(t) = \int_0^t \left|\frac{s}{t} - u\right|^{-\lambda^2} dM(s),^4 \tag{45}$$

Eq. (44) is the desired renormalization relation, which we regard as a counterpart of the star equation. We explained in Section 3 that the

[4] The integral can be interpreted as the limit of integrals as in Eq. (41). It is not hard to show based on the arguments of Mandelbrot[30] that the local Holder exponent of the sample paths of $M(t)$ defined by $A \equiv \lim_{\gamma\to 0} \log|\delta_{\gamma L} M(t)|/\log\gamma$ equals $\zeta'(q=0) = 1 + \lambda^2/2 > 1$ with probability one. It follows that the sample paths are of finite variation with probability one and pathwise integration with respect to $M(t)$ is therefore well-defined.

foundation of the star equation is a hierarchy of finite increment size approximations to the limiting distribution that quantifies its SDSI. Similarly, Eq. (38) is such a hierarchy in the scale free case of lognormal multipliers, and its zero increment limit is Eq. (44). Equation (44) is a single-time but not a single-variable equation as it is stated at a fixed time t and, therefore, quantifies the limiting distribution at that time, yet involves the Laplace transform of all $M^u(t)$. It is evident from Eq. (45) that the definition of $M^u(t)$ requires all the times up to time t. This is naturally interpreted as a reflection of the fact that nonoverlapping increments within distance L apart are not independent unlike in the canonical construction.

An essentially the same argument can be given to renormalize Eq. (40). In fact, the integral involved in Eq. (40) is the $k=N$ integral in Eq. (38) so that the argument given above goes through intact. There results

$$\frac{\partial}{\partial t}v(t,\,z)=-z\,\mathbf{E}\left[e^{-z\left(\frac{t}{L}\right)^{-\lambda^2}M^1(t)}\right]. \tag{46}$$

Equation (46) is the second renormalization relation.

As we stated in the Introduction both renormalization relations are actually valid for all gaussian processes. Although such identities are of intrinsic interest, we are particularly interested in generalizing Eq. (46) so as to interpret it as a Girsanov type formula.

We begin with a gaussian process $\omega(t)$ with covariance $c(s,\,t)$ and mean $m(t)$ and follow the steps that led to the proofs of Eqs. (44) and (46) verbatim. Equation (36) is still valid but Eq. (37) needs to be generalized as we can no longer impose the restriction $c(t,\,t)/2+m(t)=0$. Instead, we have in general

$$L(\vec{t},\,\vec{w}-\vec{k})=e^{c_{kk}/2+m_k}L(\vec{t},\,\vec{w})\,e^{-\,\vec{w}\cdot\vec{c}_k}, \tag{47}$$

where $c_{jk}=c(t_j,\,t_k)$, $m_k=m(t_k)$, and $\vec{c}_k$ has the same meaning as in Eq. (37). The extra factor propagates to Eqs. (38) and (40) so that, introducing $dM(t)\triangleq e^{\omega(t)}\,dt$, there result

$$\frac{\partial}{\partial z}v(t,\,z)=-\int_0^t e^{c(u,\,u)/2+m(u)}\,\mathbf{E}\left[e^{-z\int_0^t e^{c(s,\,u)}\,dM(s)}\right]du, \tag{48}$$

$$\frac{\partial}{\partial t}v(t,\,z)=-z\,e^{c(t,\,t)/2+m(t)}\,\mathbf{E}\left[e^{-z\int_0^t e^{c(s,\,t)}\,dM(s)}\right]. \tag{49}$$

Equations (48) and (49) look seemingly different from Eqs. (44) and (46) because we split $c(s,\,u)$ into two parts there to separate off the L dependence.

The interpretation of Eq. (49) is that for any given t there is a new probability measure such that the law of the process $s \to \omega(s) + c(s, t)$ with respect to the original measure coincides with the law of $\omega(s)$ with respect to the new measure. The details of how the new measure is defined as well as a proof of this statement are deferred to the Appendix. Suffice it to say here that in the case when $\omega(s)$ is Brownian motion, we recover the classical Girsanov theorem[13] so that our result can be interpreted as a generalized Girsanov theorem for gaussian processes.

We end this section with a short summary. The main results are Eqs. (44) and (46), which are our renormalization relations. Both quantify the SDSI of the limit process by being the zero increment limit of hierarchies of functional equations at finite increment sizes. We interpret the former and not the latter as a counterpart of the star equation because Eq. (46) involves the t derivative and therefore quantifies how the limit process changes in time. Equation (44), on the other hand, quantifies the process at a fixed time just as the star equation does in the canonical case.

8. RENORMALIZATION: APPROXIMATIONS

In this section we examine our results from a computational perspective. As we pointed out in the Introduction the main difference between Eqs. (17) and (44) is that the former involves only one unknown variable, whereas the latter involves more. The same is true of Eq. (46). For example, Eq. (44) relates the unknown Laplace transform of $M(t)$ to the unknown Laplace transforms of $M^u(t)$. On the contrary, Eq. (17), once expressed in terms of the Laplace transform of Z, involves only that transform. While it is true that Eq. (45) gives us an explicit formula for $M^u(t)$ in terms of $M(t)$, we do not know how to eliminate $M^u(t)$ from the right-hand side of Eq. (44) so as to obtain an equation involving the Laplace transform of $M(t)$ only, i.e., a single-variable equation. This is the main limitation of our work. However, we have an approximate single-variable equation in the weak intermittency limit, which is the subject of this section.

We propose the following equation in the limit $\lambda \to 0^+$

$$\frac{\partial}{\partial z} v(t, z) \approx -t \int_0^1 v\left(t, z e^{\lambda^2 (f(s) - \log \frac{t}{L})}\right) ds, \; z \geqslant 0, \; t \leqslant L, \tag{50}$$

where $f(s)$ is defined by

$$f(s) = 1 - s \log s - (1-s) \log(1-s). \tag{51}$$

Function $f(s)$ satisfies $1 \leqslant f(s) \leqslant 1 + \log(2)$ and $f(0) = f(1) = 1$. Its graph is given in Fig. 1.

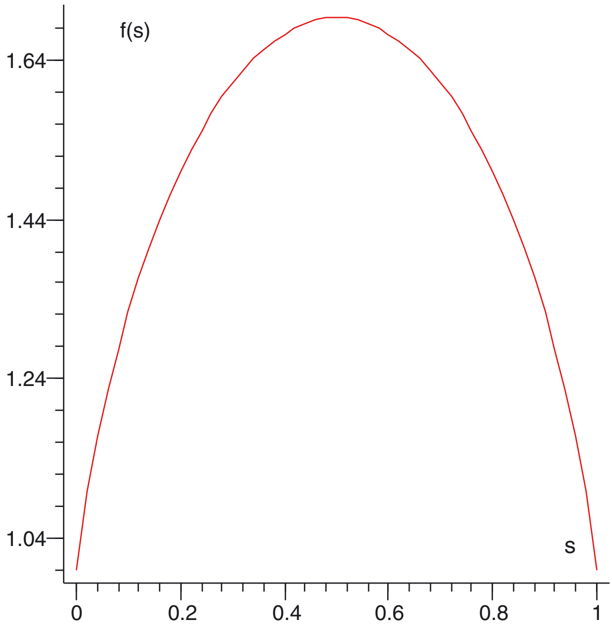

Fig. 1. Graph of $f(s)$.

It is easy to see that Eq. (50) corresponds to the first two terms of an expansion of Eq. (44) in powers of λ^2. To this end, we can write

$$M(t) = t + \lambda^2 M_{(1)}(t) + o(\lambda^2)$$

because it is obvious from Eq. (44) or otherwise that the zeroth order term of the expansion of $M(t)$ is powers of λ^2 is just t. Then, by Eq. (45)

$$M^u(t) = t + \lambda^2 \left(M_{(1)}(t) - \int_0^t \log\left|\frac{s}{t} - u\right| ds \right) + o(\lambda^2).$$

It follows that

$$\begin{aligned}
\left(\frac{t}{L}\right)^{-\lambda^2} M^u(t) &= M(t) - \lambda^2 \left(\int_0^t \log\left|\frac{s}{t} - u\right| ds + t \log\frac{t}{L} \right) + o(\lambda^2), \\
&= M(t) + t\lambda^2 \left(f(u) - \log\frac{t}{L} \right) + o(\lambda^2), \\
&= M(t) \left(1 + \lambda^2 \left[f(u) - \log\frac{t}{L} \right] \right) + o(\lambda^2).
\end{aligned}$$

Substituting this into Eq. (44), we arrive at Eq. (50), up to terms of order $o(\lambda^2)$.

Interestingly, there is more to say as to the nature of the approximation made to arrive at Eq. (50). In fact, Eq. (50) corresponds to making the substitution

$$\vec{w}\cdot\vec{c}_k \approx \lambda^2 f\left(\frac{k}{N}\right)\sum_{j=1}^{N} w_j \tag{52}$$

in Eq. (38). In other words, if we make this substitution, the $\Delta t \to 0$ limit of Eq. (38) becomes straightforward and we obtain Eq. (50). Space limitations preclude us from giving a detailed argument explaining the origin of the substitution, suffice it to say that Eq. (52) comes from the $N \to \infty$ asymptotic of the eigenvalues and eigenvectors of the covariance matrix C given by Eqs. (30) and (31).

It remains to discuss the accuracy of our approximation. It is more interesting to examine the characteristic function $\phi(t, q)$ of $M(t)$ as opposed to the Laplace transform $v(t, z)$. The two are formally related *via* $\phi(t, q) = v(t, z)$ for $q = iz$ and $z \geqslant 0$. There results

$$\frac{\partial}{\partial q}\phi(t, q) \approx it \int_0^1 \phi(t, q e^{\lambda^2 (f(s) - \log \frac{t}{L})}) ds. \tag{53}$$

We tested Eq. (53) numerically for $\lambda = 0.5$ and $t = L$ by independently simulating $\phi(L, q)$ and its derivative and substituting these into the equation. Results are exhibited in Figs. 2 and 3. We independently tested Eq. (53) by, first, precomputing the pdf of $M(L)$ and, then, computing $\phi(L, q)$ and its derivative by quadrature. Results are in Figs. 4–6. Both tests indicate that Eq. (53) describes the characteristic function of the distribution quite accurately for λ as large as 0.5.

9. CONCLUSIONS

We presented a new framework for analyzing the exponential functional of a gaussian process. The gist of our approach is to consider a hierarchy of finite increment size approximations to the Laplace transform of the limit process and represent these approximations as multiple contour integrals. The hierarchy is a mathematical means of quantifying the SDSI of the process. Such representations, the special case of which was first introduced by Leipnik[23] in the context of lognormal random

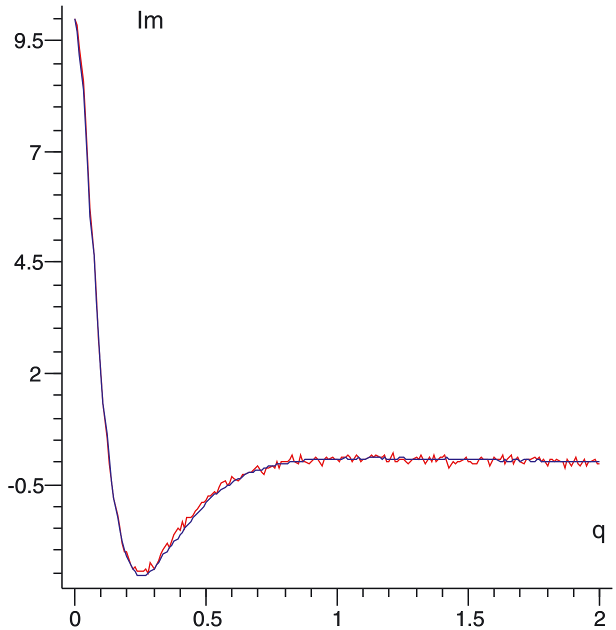

Fig. 2. Imaginary part of the left-hand side (red curve) and right-hand side (blue curve) of Eq. (53). $\phi(L, q)$ and its q derivative were each computed by simulating 20000 sample paths of length $N=256$. $L=10$, $\lambda=0.5$.

variables, are quite general and apply to all processes built as the exponential functional of an underlying process as seen from Eq. (25). As an elementary application of Eq. (25) we derived a Feynman-Kac formula for the Laplace transform of the limiting distribution of the limit lognormal multifractal[27,3] and showed that it does not capture the distribution uniquely. We also showed how our discrete scale approximations can be used in principle to establish Feynman-Kac formulae for general gaussian processes.

In the case of a single lognormal random variable Leipnik[23] found an integro-differential equation for the Laplace transform, Eq. (22). We introduced the notion of renormalization in the sense of Leipnik as an extension of Leipnik's equation to the general exponential functionals. By that we mean integro-differential relations that express the z and t derivatives of the Laplace transform as averages of rescaled Laplace transforms of possibly different but related distributions. At the level of finite scale approximations there resulted Eq. (36) valid for all limit log-infinitely divisible processes. When the underlying process is gaussian, we found exact renormalization relations for the limit process. In particular, in the

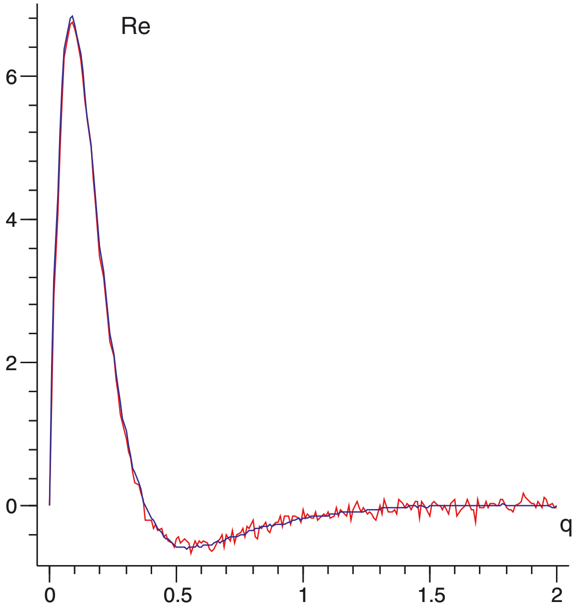

Fig. 3. Real part of the left-hand side (red curve) and right-hand side (blue curve) of Eq. (53). $\phi(L, q)$ and its q derivative were each computed by simulating 20000 sample paths of length $N = 256$. $L = 10$, $\lambda = 0.5$.

case of the limit lognormal multifractal the z relation is interpreted as a counterpart of the star equation of Mandelbrot[28,29] and the t relation shown to be a Girsanov type formula. The first interpretation is based on our understanding of the star equation as the limit of a hierarchy of finite increment size approximations to the limit process that quantifies its SDSI. The hierarchy constructed in this paper shares the same features and, moreover, the resulting renormalization relation is stated at a fixed time t as is the star equation. The t relation provides another way of quantifying the SDSI and reflects how the process changes in time. Mathematically, the t relation is equivalent to a generalized Girsanov theorem, i.e., reflects an invariance of the underlying gaussian process under a change of measure, and recovers the classical Girsanov theorem[13] for Brownian motion.

Our relations express the z and t derivatives of the Laplace transform as averages of rescaled Laplace transforms over a family of limit lognormal multifractals derived from the original limit process. As a result these relations cannot be regarded as single-variable equations. This is the main limitation of our work. In the weak intermittency limit we presented an approximate single-variable equation for the Laplace transform.

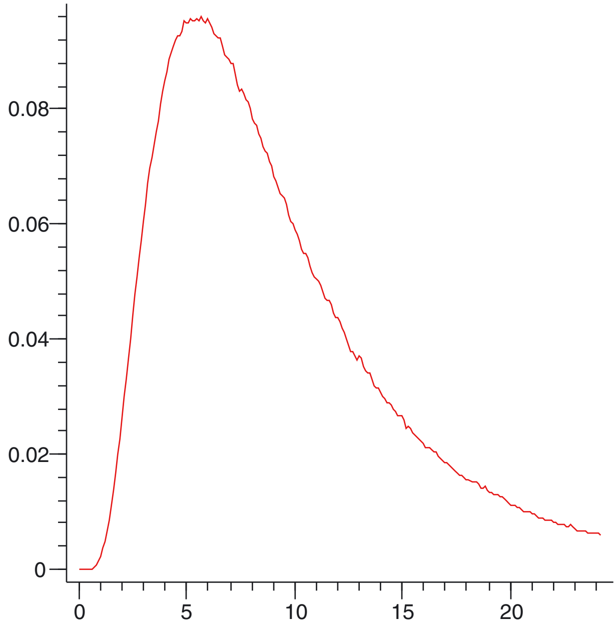

Fig. 4. Probability density function of $M(L)$ computed via a kernel density estimator by simulating 1000000 sample paths of length $N=256$. $L=10$, $\lambda=0.5$.

We end by describing a number of open problems and new directions. First and foremost, it is an open problem to fully explain the information that is contained in Eqs. (44) and (46). This task is not easy precisely because these are not single-variable equations, which is a direct reflection of a much more complicated dependence structure than it is in the canonical case. Our treatment of the weak intermittency limit indicates that it may be possible to obtain an exact single-variable integro-differential equation for the Laplace transform of $M(t)$ by expanding Eq. (44) in powers of λ^2 to *all* orders. The difficulty of such an expansion is that Eq. (44) involves the whole path of the process up to time t as opposed to just its value at t. The hardest question in this direction is as to the functional form of the resulting equation. If such an equation exists, Eq. (50) gives the first nontrivial term of its expansion in powers of λ^2. Perhaps a simpler question would be to ask if Eq. (44) determines the limit distribution uniquely. A similar question is whether multifractality in the sense of Eq. (2) *and* stationary increments capture the limit uniquely.

Second, there are many subtle properties of the canonical multifractals such as conditions for the existence of negative moments, for example, that were discovered by analyzing the star equation(5,14,22,25). The same

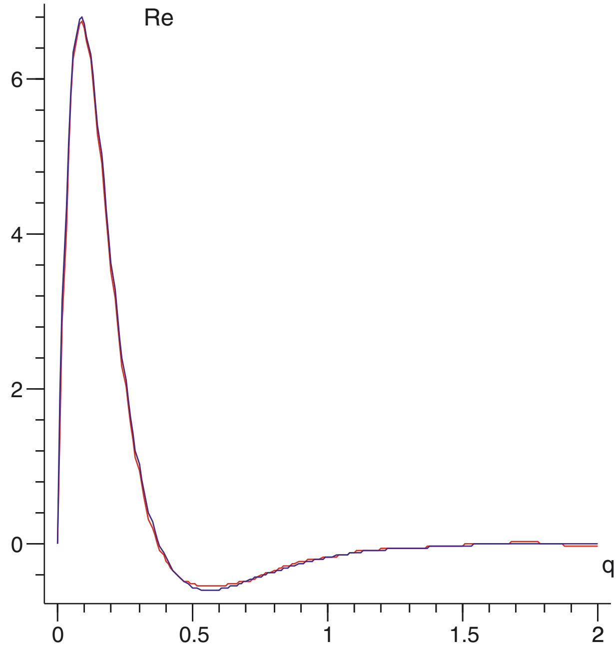

Fig. 5. Real part of the left-hand side (red curve) and right-hand side (blue curve) of Eq. (53). $\phi(L, q)$ and its q derivative were computed by quadrature using the simulated pdf of $M(L)$.

sort of questions arises naturally in the limit lognormal case. For example, Fig. 4 strongly suggests that $M(t)$ has finite negative moments but this appears very hard to prove with the tools that we have. The primary difficulty is the lack of knowledge of how the asymptotics of $M^u(t)$ relate to those of $M(t)$. The same sort of knowledge is also needed to compute the asymptotic of the left tail of the pdf of $M(t)$.

Finally, an even more challenging problem is how to renormalize in the general case of Eq. (36), i.e., not assuming lognormality. The problem requires an in-depth analysis of $L(\vec{t}, \vec{w})$ and might turn out to be tractable, at least for those cases when Eq. (26) holds true, as there is an explicit formula for $Q(\vec{t}, \vec{w})$ stated in ref. 4.

APPENDIX

Our goal is to formalize and prove the statement made in Section 7 regarding the interpretation of Eq. (49) as a generalized Girsanov theorem.

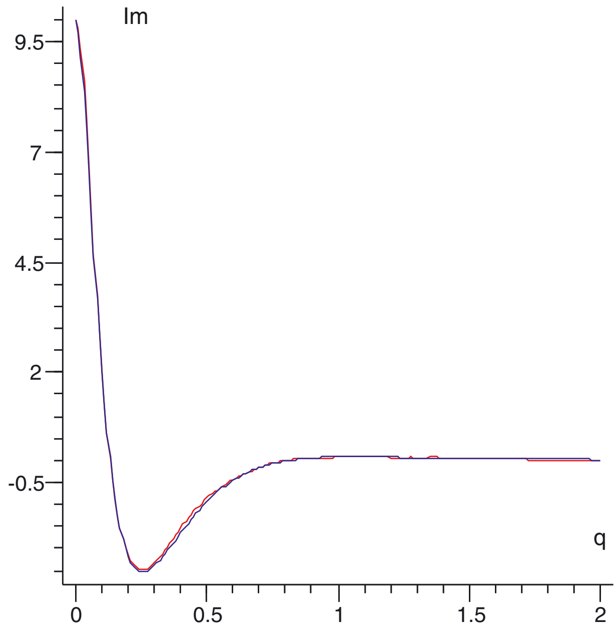

Fig. 6. Imaginary part of the left-hand side (red curve) and right-hand side (blue curve) of Eq. (53). $\phi(L, q)$ and its q derivative were computed by quadrature using the simulated pdf of $M(L)$.

Let $\mathcal{P}$ be the original probability measure associated with the gaussian process $\omega(s)$, i.e., the measure with respect to which we take expectation to define $v(t, z)$ as well as compute the right-hand side of Eqs. (48) and (49). Given an arbitrary time $t > 0$, we define a new measure $\mathcal{Q}$ equivalent to $\mathcal{P}$, i.e., having the same null sets, as follows.

$$d\mathcal{Q} \triangleq \mathcal{E}\, d\mathcal{P},$$

where $\mathcal{E}$ is a random variable defined by

$$\mathcal{E} \triangleq e^{\omega(t) - m(t) - \frac{1}{2}c(t,t)}.$$

It is obvious that $\mathcal{E}$ is positive with the $\mathcal{P}$ expectation equal to 1 so that $\mathcal{Q}$ is indeed a probability measure. Then, the formal statement of our result is that Eq. (49) is equivalent to the following proposition.

The law of the process $s \to \omega(s) + c(s, t)$, $s \leqslant t$, with respect to the original measure $\mathcal{P}$ is the same as the law of the original process $\omega(s)$ with respect to the new measure $\mathcal{Q}$.

We note that when $\omega(s)$ is Brownian motion, then $\omega(s) + c(s, t) = B(s) + s$, $s \leqslant t$, and we indeed recover the classical result[13].

Proof that Eq. (49) Implies the Proposition. Since Eq. (49) is valid for all gaussian processes, we apply it to the process $\widetilde{\omega}(s) \triangleq \omega(s) + \log(g(s))$, where $g(s)$ is an arbitrary smooth and positive function. Note that $\widetilde{\omega}(s)$ has the same covariance structure and $\mathcal{E}$ as those associated with $\omega(s)$, whereas its mean is $\widetilde{m}(s) = m(s) + \log(g(s))$. Consider $\mathbf{E}_{\mathcal{Q}}\left[e^{-z\int_0^t e^{\widetilde{\omega}(s)}\,ds}\right]$ and change the measure to $\mathcal{P}$.

$$\begin{aligned}\mathbf{E}_{\mathcal{Q}}\left[e^{-z\int_0^t e^{\widetilde{\omega}(s)}\,ds}\right] &= \mathbf{E}_{\mathcal{P}}\left[e^{-z\int_0^t e^{\widetilde{\omega}(s)}\,ds}\,\mathcal{E}\right],\\ &= \frac{e^{-\widetilde{m}(t)-\frac{1}{2}c(t,t)}}{-z}\frac{\partial}{\partial t}\mathbf{E}_{\mathcal{P}}\left[e^{-z\int_0^t e^{\widetilde{\omega}(s)}\,ds}\right].\end{aligned}$$

It follows by Eq. (49) applied to $\widetilde{\omega}(s)$ that

$$\mathbf{E}_{\mathcal{Q}}\left[e^{-z\int_0^t e^{\widetilde{\omega}(s)}\,ds}\right] = \mathbf{E}_{\mathcal{P}}\left[e^{-z\int_0^t e^{\widetilde{\omega}(s)+c(s,t)}\,ds}\right].$$

The Laplace transform is injective, therefore, we proved that the law of $\int_0^t e^{\omega(s)+c(s,t)} g(s)\,ds$ with respect to $\mathcal{P}$ coincides with the law of $\int_0^t e^{\omega(s)} g(s)\,ds$ with respect to $\mathcal{Q}$. Finally, since $g(s)$ is arbitrary, we arrive at the equality in distribution of the integrands, as desired. ∎

Proof of the Converse. We showed above that

$$\frac{\partial}{\partial t}\mathbf{E}_{\mathcal{P}}\left[e^{-z\int_0^t e^{\omega(s)}\,ds}\right] = -z e^{m(t)+\frac{1}{2}c(t,t)}\mathbf{E}_{\mathcal{Q}}\left[e^{-z\int_0^t e^{\omega(s)}\,ds}\right]$$

follows from the definition of $\mathcal{E}$. Combining this equation with the assumption hypothesis

$$\mathbf{E}_{\mathcal{Q}}\left[e^{-z\int_0^t e^{\omega(s)}\,ds}\right] = \mathbf{E}_{\mathcal{P}}\left[e^{-z\int_0^t e^{\omega(s)+c(s,t)}\,ds}\right],$$

we arrive at Eq. (49). ∎

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